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# Extracting the cosmological constant from the Wheeler–DeWitt equation in a modified gravity theory

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## Abstract

We discuss how to extract information about the cosmological constant from the Wheeler–DeWitt equation, considered as an eigenvalue of a Sturm–Liouville problem. A generalization to a  $f(R)$  theory is taken under examination. The equation is approximated to one loop with the help of a variational approach with Gaussian trial wave functionals. We use a zeta function regularization to handle with divergences. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation.

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## 1. Introduction

Einstein's field equations represent a fundamental set of information regarding the laws of spacetime. They are represented by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_c g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (1)$$

where  $T_{\mu\nu}$  is the energy–momentum tensor of some matter fields,  $\kappa = 8\pi G$  with  $G$  as Newton's constant and  $\Lambda_c$  is the cosmological constant. The sourceless version of equation (1) is simply

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_c g_{\mu\nu} = 0. \quad (2)$$

It is well known that there exists a huge discrepancy between the observed [1] and the computed value of the cosmological constant. It amounts approximately to a factor of 120 orders of magnitude: this is the *cosmological constant problem*. One possible approach to such a problem is given by the Wheeler–DeWitt (WDW) equation [2]. The WDW equation can be extracted from Einstein's field equations with and without matter fields in a very simple way.

If we introduce a time-like unit vector  $u^\mu$  such that  $u \cdot u = -1$ , then after a little rearrangement, we get<sup>1</sup>

$$\mathcal{H} = (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}({}^3R - 2\Lambda_c) = 0. \tag{3}$$

${}^3R$  is the scalar curvature in three dimensions. This is the time–time component of equation (2). It represents a constraint at the classical level and the invariance under *time* reparametrization. Its quantum counterpart

$$\mathcal{H}\Psi = 0 \tag{4}$$

is the WDW equation. If we integrate over the hypersurface  $\Sigma$  and we define

$$\hat{\Lambda}_\Sigma = (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}{}^3R, \tag{5}$$

equation (4) can be cast in the following form:

$$\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}] \left[ \int_\Sigma d^3x \hat{\Lambda}_\Sigma \right] \Psi[g_{ij}] = -\frac{\Lambda_c}{\kappa} V \int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\Psi[g_{ij}], \tag{6}$$

where we have multiplied equation (4) by  $\Psi^*[g_{ij}]$ , we have functionally integrated over the three spatial metric  $g_{ij}$  and where we have defined the volume of the hypersurface  $\Sigma$  as  $V = \int_\Sigma d^3x \sqrt{g}$ . Thus one can formally re-write the WDW equation as

$$\frac{1}{V} \frac{\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}] \int_\Sigma d^3x \hat{\Lambda}_\Sigma \Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\Psi[g_{ij}]} = \frac{1}{V} \frac{\langle \Psi | \int_\Sigma d^3x \hat{\Lambda}_\Sigma | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda_c}{\kappa}. \tag{7}$$

We can gain more information considering a separation of the spatial part of the metric into a background term,  $\bar{g}_{ij}$ , and a quantum fluctuation,  $h_{ij}$ ,

$$g_{ij} = \bar{g}_{ij} + h_{ij}. \tag{8}$$

Thus equation (7) becomes

$$\frac{1}{V} \frac{\langle \Psi | \int_\Sigma d^3x [\hat{\Lambda}_\Sigma^{(0)} + \hat{\Lambda}_\Sigma^{(1)} + \hat{\Lambda}_\Sigma^{(2)} + \dots] | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda_c}{\kappa} \Psi[g_{ij}], \tag{9}$$

where  $\hat{\Lambda}_\Sigma^{(i)}$  represents the  $i$ th order of perturbation in  $h_{ij}$ . By observing that the kinetic part of  $\hat{\Lambda}_\Sigma$  is quadratic in the momenta, we only need to expand the three-scalar curvature  $\int d^3x \sqrt{g} {}^3R$  up to the quadratic order and we get

$$\int_\Sigma d^3x \sqrt{g} \left[ -\frac{1}{4}h\Delta h + \frac{1}{4}h^{li}\Delta h_{li} - \frac{1}{2}h^{ij}\nabla_l\nabla_i h^l_j + \frac{1}{2}h\nabla_l\nabla_i h^{li} - \frac{1}{2}h^{ij}R_{ia}h^a_j + \frac{1}{2}hR_{ij}h^{ij} + \frac{1}{4}h(R^{(0)}h) \right] \tag{10}$$

where  $h$  is the trace of  $h_{ij}$  and  $R^{(0)}$  is the three-dimensional scalar curvature on-shell. Equation (7) represents the Sturm–Liouville problem associated with the cosmological constant. The related boundary conditions are dictated by the choice of the trial wavefunctionals which, in our case, are of the Gaussian type. Different types of wavefunctionals correspond to different boundary conditions. Extracting the TT tensor contribution from equation (7) approximated to second order in perturbation of the spatial part of the metric into a background term,  $\bar{g}_{ij}$ , and a perturbation,  $h_{ij}$ , we get

$$\hat{\Lambda}_\Sigma^\perp = \frac{1}{4V} \int_\Sigma d^3x \sqrt{g} G^{ijkl} \left[ (2\kappa)K^{-1\perp}(x, x)_{ijkl} + \frac{1}{(2\kappa)}(\Delta_2)_j^a K^\perp(x, x)_{iakl} \right], \tag{11}$$

<sup>1</sup> See [3] for more details.

where  $G^{ijkl}$  represents the inverse DeWitt metric and all indices run from one to three. The propagator  $K^\perp(x, x)_{iakl}$  can be represented as

$$K^\perp(\vec{x}, \vec{y})_{iakl} := \sum_\tau \frac{h_{ia}^{(\tau)\perp}(\vec{x})h_{kl}^{(\tau)\perp}(\vec{y})}{2\lambda(\tau)}, \quad (12)$$

where  $h_{ia}^{(\tau)\perp}(\vec{x})$  are the eigenfunctions of  $\Delta_2$ , whose explicit expression for the massive case will be shown in the following section.  $\tau$  denotes a complete set of indices and  $\lambda(\tau)$  are a set of variational parameters to be determined by the minimization of equation (11). The expectation value of  $\hat{\Lambda}_2^\perp$  is easily obtained by inserting the form of the propagator into equation (11) and minimizing with respect to the variational function  $\lambda(\tau)$ . Thus the total one-loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{4} \sum_\tau [\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}]. \quad (13)$$

The above expression makes sense only for  $\omega_i^2(\tau) > 0$ , where  $\omega_i$  are the eigenvalues of  $\Delta_2$ .

## 2. One loop energy regularization and renormalization for a $f(R) = R$ theory

The spin-2 operator for the Schwarzschild metric in the Regge and Wheeler representation [4], leads to the following system of equations ( $r \equiv r(x)$ ):

$$\begin{cases} \left[ -\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_1^2(r) \right] f_1(x) = \omega_{1,l}^2 f_1(x) \\ \left[ -\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_2^2(r) \right] f_2(x) = \omega_{2,l}^2 f_2(x), \end{cases} \quad (14)$$

where reduced fields have been used and the proper geodesic distance from the *throat* of the bridge has been considered. Close to the throat, the effective masses are

$$\begin{cases} m_1^2(r) \simeq -m_0^2(M) \\ m_2^2(r) \simeq m_0^2(M), \end{cases} \quad (15)$$

where we have defined a parameter  $r_0 > 2MG$  and  $m_0^2(M) = 3MG/r_0^3$ . The main reason for introducing a new parameter resides in the fluctuation of the horizon that forbids any kind of approach. It is now possible to explicitly evaluate equation (13) in terms of the effective mass. To further proceed we use the WKB method used by 't Hooft in the brick wall problem [5] and we count the number of modes with frequency less than  $\omega_i$ ,  $i = 1, 2$ . Thus the one-loop total energy for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = \rho_1 + \rho_2 = -\frac{1}{16\pi^2} \sum_{i=1}^2 \int_{\sqrt{m_i^2(r)}}^{+\infty} \omega_i^2 \sqrt{\omega_i^2 - m_i^2(r)} d\omega_i, \quad (16)$$

where we have included an additional  $4\pi$  coming from the angular integration. Here, we use the zeta function regularization method to compute the energy densities  $\rho_1$  and  $\rho_2$ . Note that this procedure is completely equivalent to the subtraction procedure of the Casimir energy computation where the zero point energy (ZPE) in different backgrounds with the same asymptotic properties is involved. To this purpose, we introduce the additional mass parameter  $\mu$  in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. One gets

$$\rho_i(\varepsilon) = \frac{m_i^4(r)}{256\pi^2} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{\mu^2}{m_i^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right], \quad (17)$$

$i = 1, 2$ . The renormalization is performed via the absorption of the divergent part into the re-definition of the bare classical constant  $\Lambda$

$$\Lambda \rightarrow \Lambda_0 + \Lambda^{\text{div}}. \tag{18}$$

The remaining finite value for the cosmological constant reads

$$\begin{aligned} \frac{\Lambda_0}{8\pi G} = \frac{1}{256\pi^2} \left\{ m_1^4(r) \left[ \ln \left( \frac{\mu^2}{|m_1^2(r)|} \right) + 2 \ln 2 - \frac{1}{2} \right] \right. \\ \left. + m_2^4(r) \left[ \ln \left( \frac{\mu^2}{m_2^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right] \right\} = (\rho_1(\mu) + \rho_2(\mu)) = \rho_{\text{eff}}^{\text{TT}}(\mu, r). \end{aligned} \tag{19}$$

The quantity in equation (19) depends on the arbitrary mass scale  $\mu$ . It is appropriate to use the renormalization group equation to eliminate such a dependence. To this end, we impose that [6]

$$\frac{1}{8\pi G} \mu \frac{\partial \Lambda_0^{\text{TT}}(\mu)}{\partial \mu} = \mu \frac{d}{d\mu} \rho_{\text{eff}}^{\text{TT}}(\mu, r). \tag{20}$$

Solving it we find that the renormalized constant  $\Lambda_0$  should be treated as a running one in the sense that it varies provided that the scale  $\mu$  is changing

$$\Lambda_0(\mu, r) = \Lambda_0(\mu_0, r) + \frac{G}{16\pi} (m_1^4(r) + m_2^4(r)) \ln \frac{\mu}{\mu_0}. \tag{21}$$

Substituting equation (21) into equation (19) we find

$$\frac{\Lambda_0(\mu_0, M)}{8\pi G} = -\frac{1}{128\pi^2} \left\{ m_0^4(M) \left[ \ln \left( \frac{m_0^2(M)}{4\mu_0^2} \right) + \frac{1}{2} \right] \right\}. \tag{22}$$

Equation (22) has a maximum when<sup>2</sup>

$$\frac{1}{e} = \frac{m_0^2(M)}{4\mu_0^2} \implies \Lambda_0(\mu_0) = \frac{Gm_0^4(M)}{32\pi} = \frac{G\mu_0^4}{2\pi e^2}. \tag{23}$$

The computed cosmological constant appears to depend on the Schwarzschild radius. This dependence simply reflects the fact that the chosen background introduces one physical scale: the Schwarzschild radius. Nothing prevents us from considering a more general situation where the scalar curvature  $R$  is replaced by a generic function of  $R$ . Therefore, we will consider the Sturm–Liouville problem of equation (7) in the context of a  $f(R)$  theory<sup>3</sup>.

### 3. One loop energy regularization and renormalization for a generic $f(R)$ theory in a Hamiltonian formulation

In this section, we report the main steps discussed in [9] for a  $f(R)$  theory in connection with the Sturm–Liouville problem of equation (7). Although a  $f(R)$  theory does not need a cosmological constant, rather it should explain it, we shall consider the following Lagrangian density describing a generic  $f(R)$  theory of gravity

$$\mathcal{L} = \sqrt{-g}(f(R) - 2\Lambda), \quad \text{with} \quad f'' \neq 0, \tag{24}$$

where  $f(R)$  is an arbitrary smooth function of the scalar curvature and primes denote differentiation with respect to the scalar curvature. A cosmological term is added also in

<sup>2</sup> Remark. Note that in any case, the maximum of  $\Lambda$  corresponds to the minimum of the energy density.

<sup>3</sup> A recent review on the problem of  $f(R)$  theories can be found in [7]. A more general discussion on modified gravities of the type  $f(R)$ ,  $f(G)$  and  $f(R, G)$  where  $G$  is the Gauss–Bonnet invariant, can be found in [8].

this case for the sake of generality, because in any case, equation (24) represents the most general Lagrangian to examine. Obviously  $f'' = 0$  corresponds to GR. The generalized Hamiltonian density for the  $f(R)$  theory assumes the form<sup>4</sup>

$$\mathcal{H} = \frac{1}{2\kappa}[-\sqrt{g}f'(R)(^{(3)}R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2) + V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij} - 2p^{ij}K_{ij}], \quad (25)$$

where

$$\mathcal{P} = -6\sqrt{g}f'(R) \quad (26)$$

and

$$V(\mathcal{P}) = \sqrt{g}[Rf'(R) - f(R)]. \quad (27)$$

Henceforth, the superscript 3 indicating the spatial part of the metric will be omitted on the metric itself. When  $f(R) = R$ ,  $V(\mathcal{P}) = 0$  as it should be. Equation (25) becomes

$$\begin{aligned} \mathcal{H} = f'(R) & \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}(^{(3)}R - 2\Lambda_c) \right] \\ & + \frac{1}{2\kappa} \left[ \sqrt{g}f'(R)(2K_{ij}K^{ij}) + V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij} - 2p^{ij}K_{ij} \right]. \end{aligned} \quad (28)$$

Since

$$p^{ij} = \sqrt{g}K^{ij}, \quad (29)$$

then we obtain

$$\begin{aligned} \mathcal{H} = f'(R) & \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}(^{(3)}R - 2\Lambda_c) \right] \\ & + \frac{1}{2\kappa} [2\sqrt{g}K_{ij}K^{ij}(f'(R) - 1) + V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij}] \end{aligned} \quad (30)$$

and transforming into canonical momenta, one gets

$$\begin{aligned} \mathcal{H} = f'(R) & \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}(^{(3)}R - 2\Lambda_c) \right] \\ & + 2(2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] (f'(R) - 1) + \frac{1}{2\kappa} [V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij}]. \end{aligned} \quad (31)$$

By imposing the Hamiltonian constraint, we obtain

$$\begin{aligned} f'(R) & \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}(^{(3)}R) \right] + 2(2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] (f'(R) - 1) \\ & + \frac{1}{2\kappa} [V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij}] = -f'(R)\sqrt{g}\frac{\Lambda_c}{\kappa}. \end{aligned} \quad (32)$$

If we assume that  $f'(R) \neq 0$  the previous expression becomes

$$\begin{aligned} & \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}(^{(3)}R) \right] + (2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] \frac{2(f'(R) - 1)}{f'(R)} \\ & + \frac{1}{2\kappa f'(R)} [V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij}] = -\sqrt{g}\frac{\Lambda_c}{\kappa}. \end{aligned} \quad (33)$$

<sup>4</sup> See [10] for technical details.

Now, we integrate over the hypersurface  $\Sigma$  to obtain

$$\int_{\Sigma} d^3x \left\{ \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}^{(3)}}{2\kappa} R \right] + (2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] \frac{2(f'(R) - 1)}{f'(R)} \right\} \quad (34)$$

$$+ \int_{\Sigma} d^3x \frac{1}{2\kappa f'(R)} [V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{|ij}] = -\frac{\Lambda_c}{\kappa} \int_{\Sigma} d^3x \sqrt{g}. \quad (35)$$

The term

$$\frac{1}{\kappa} \int_{\Sigma} d^3x \frac{1}{f'(R)} g^{ij}(\sqrt{g}f'(R))_{|ij} \quad (36)$$

appears to be a three-divergence and therefore will not contribute to the computation. The remaining equation simplifies into

$$\int_{\Sigma} d^3x \left\{ \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}^{(3)}}{2\kappa} R \right] + (2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] \frac{2(f'(R) - 1)}{f'(R)} + \frac{V(\mathcal{P})}{2\kappa f'(R)} \right\} = -\frac{\Lambda_c}{\kappa} \int_{\Sigma} d^3x \sqrt{g}. \quad (37)$$

By a canonical procedure of quantization, we want to obtain the vacuum state of a generic  $f(R)$  theory. By repeating the same procedure for the generalized WDW equation (37), we obtain

$$\frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x [\hat{\Lambda}_{\Sigma}^{(2)}] | \Psi \rangle}{\langle \Psi | \Psi \rangle} + \frac{2\kappa}{V} \frac{2(f'(R) - 1)}{f'(R)} \frac{\langle \Psi | \int_{\Sigma} d^3x [G_{ijkl}\pi^{ij}\pi^{kl} + \pi^2/4] | \Psi \rangle}{\langle \Psi | \Psi \rangle} + \frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x V(\mathcal{P}) / (2\kappa f'(R)) | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda_c}{\kappa}. \quad (38)$$

From equation (38), we can define a ‘modified’  $\hat{\Lambda}_{\Sigma}^{(2)}$  operator which includes  $f'(R)$ . Thus, we obtain

$$\frac{\langle \Psi | \int_{\Sigma} d^3x [\hat{\Lambda}_{\Sigma, f(R)}^{(2)}] | \Psi \rangle}{\langle \Psi | \Psi \rangle} + \frac{\kappa}{V} \frac{(f'(R) - 1)}{f'(R)} \frac{\langle \Psi | \int_{\Sigma} d^3x [\pi^2] | \Psi \rangle}{\langle \Psi | \Psi \rangle} + \frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x \frac{V(\mathcal{P})}{2\kappa f'(R)} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda_c}{\kappa}, \quad (39)$$

where

$$\hat{\Lambda}_{\Sigma, f(R)}^{(2)} = (2\kappa)h(R)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}^3}{2\kappa} R^{\text{lin}}, \quad (40)$$

with

$$h(R) = 1 + \frac{2[f'(R) - 1]}{f'(R)} \quad (41)$$

and where  ${}^3R^{\text{lin}}$  is the linearized scalar curvature whose expression is shown in square brackets in equation (10). Note that when  $f(R) = R$ , consistently it is  $h(R) = 1$ . From equation (39), we redefine  $\Lambda_c$

$$\Lambda'_c = \Lambda_c + \frac{1}{2V} \frac{\langle \Psi | \int_{\Sigma} d^3x \frac{V(\mathcal{P})}{f'(R)} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \Lambda_c + \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)}, \quad (42)$$

where we have explicitly used the definition of  $V(\mathcal{P})$ . In the same spirit as the previous section, we restrict the analysis to the contribution of physical degrees of freedom, namely TT tensors<sup>5,6</sup>. Thus equation (22) becomes

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = -\frac{m_0^4(M)}{128\pi^2} \left[ \ln \left( \frac{m_0^2(M)}{4\mu_0^2} \right) + \frac{1}{2} \right]. \quad (43)$$

Now, we compute the maximum of  $\Lambda'_0$ , by setting  $x = m_0^2(M)/4\mu_0^2$ . Thus  $\Lambda'_0$  becomes

$$\Lambda'_0(\mu_0, x) = -\frac{G\mu_0^4}{\pi} x^2 \left[ \ln(x) + \frac{1}{2} \right]. \quad (44)$$

As a function of  $x$ ,  $\Lambda_0(\mu_0, x)$  vanishes for  $x = 0$  and  $x = \exp(-\frac{1}{2})$  and when  $x \in [0, \exp(-\frac{1}{2})]$ ,  $\Lambda'_0(\mu_0, x) \geq 0$ . It has a maximum for  $\bar{x} = 1/e$  equivalent to  $m_0^2(M) = 4\mu_0^2/e$  and its value is

$$\Lambda'_0(\mu_0, \bar{x}) = \frac{G\mu_0^4}{2\pi e^2} \quad (45)$$

or

$$\frac{1}{\sqrt{h(R)}} \left[ \Lambda_0(\mu_0, \bar{x}) + \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)} \right] = \frac{G\mu_0^4}{2\pi e^2}. \quad (46)$$

Isolating  $\Lambda_0(\mu_0, \bar{x})$ , we get

$$\Lambda_0(\mu_0, \bar{x}) = \sqrt{h(R)} \frac{G\mu_0^4}{2\pi e^2} - \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)}. \quad (47)$$

Note that  $\Lambda_0(\mu_0, \bar{x})$  can be set to zero when

$$\sqrt{h(R)} \frac{G\mu_0^4}{2\pi e^2} = \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)}. \quad (48)$$

Let us see what happens when  $f(R) = \exp(-\alpha R)$ . This choice is simply suggested by the regularity of the function at every scale. In this case, equation (48) becomes

$$\sqrt{\frac{3\alpha \exp(-\alpha R) + 2}{\alpha \exp(-\alpha R)}} \frac{G\mu_0^4}{\pi e^2} = \frac{1}{\alpha V} \int_{\Sigma} d^3x \sqrt{g} (1 + \alpha R). \quad (49)$$

For Schwarzschild, it is  $R = 0$ , and by setting  $\alpha = G$ , we have the relation

$$\mu_0^4 = \frac{\pi e^2}{G} \sqrt{\frac{1}{(3G + 2)G}}. \quad (50)$$

Like the case of  $f(R) = R$ , the purpose of this calculation is related to equation (7), which can be applied to different backgrounds, i.e. de Sitter, Schwarzschild–de Sitter, etc. The result of this process will be a spectrum built in terms of vacuum fluctuations. Once the ‘ground state’ of this spectrum will be identified, then there will be a chance to have an approach for explaining the so-called *dark energy*. Note that in this approach, there is no evolution in time, so some pathologies of the type ‘*big rip*’ do not come into play at this stage.

<sup>5</sup> For a complete derivation of the effective action for a  $f(R)$  theory, see [11].

<sup>6</sup> By a canonical decomposition of the gauge part  $\xi_a$  into a transverse part  $\xi_a^T$  with  $\nabla^a \xi_a^T = 0$  and a longitudinal part  $\xi_a^{\parallel}$  with  $\xi_a^{\parallel} = \nabla_a \psi$ , it is possible to show that most of the contribution comes from the longitudinal part (scalar). Evidence against scalar perturbation contribution in a Schwarzschild background has been discussed in [12].



## References

- [1] For a pioneering review on this problem see Weinberg S 1989 *Rev. Mod. Phys.* **61** 1  
For more recent and detailed reviews see Sahni V and Starobinsky A 2000 *Int. J. Mod. Phys. D* **9** 373 (Preprint [astro-ph/9904398](#))  
Straumann N 2002 The history of the cosmological constant problem Preprint [gr-qc/0208027](#)  
Padmanabhan T 2003 *Phys. Rept.* **380** 235 (Preprint [hep-th/0212290](#))
- [2] DeWitt B S 1967 *Phys. Rev.* **160** 1113
- [3] Garattini R 2006 *J. Phys. A: Math. Gen.* **39** 6393 (Preprint [gr-qc/0510061](#))  
Garattini R 2006 *J. Phys. Conf. Ser.* **33** 215 (Preprint [gr-qc/0510062](#))
- [4] Regge T and Wheeler J A 1957 *Phys. Rev.* **108** 1063
- [5] 't Hooft G 1985 *Nucl. Phys. B* **256** 727
- [6] Perez-Mercader J and Odintsov S D 1992 *Int. J. Mod. Phys. D* **1** 401  
Cherednikov I O 2002 *Acta Physica Slovaca* **52** 221  
Cherednikov I O 2004 *Acta Phys. Polon. B* **35** 1607  
Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1  
Inclusion of non-perturbative effects, namely beyond one-loop, in de Sitter quantum gravity have been discussed in Falkenberg S and Odintsov S D 1998 *Int. J. Mod. Phys. A* **13** 607 (Preprint [hep-th/9612019](#))
- [7] Capozziello S and Francaviglia M 2007 Extended theories of gravity and their cosmological and astrophysical applications *Gen. Rel. Grav.* [at press](#) (Preprint [0706.1146](#))
- [8] Nojiri S and Odintsov S D 2007 *Int. J. Geom. Meth. Mod. Phys.* **4** 115 (Preprint [hep-th/0601213](#))
- [9] Capozziello S and Garattini R 2007 *Class. Quant. Grav.* **24** 1627 (Preprint [gr-qc/0702075](#))
- [10] Querella L 1999 Variational principles and cosmological models in higher-order gravity *PhD Thesis* (Preprint [gr-qc/9902044](#))
- [11] Cognola G, Elizalde E, Nojiri S, Odintsov S D and Zerbini S 2005 *J. Cosmol. Astropart. Phys.* **JCAP02(2005)10** (Preprint [hep-th/0501096](#))
- [12] Garattini R 2004 *TSPU Vestnik* **44** N7 72 (Preprint [gr-qc/0409016](#))
- [13] Abdalla M C B, Nojiri S and Odintsov S D 2005 *Class. Quantum Grav.* **22** L35 (Preprint [hep-th/0409177](#))